Math C067 Bayes' Theorem

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Gold and Silver Balls. Bill has three boxes. Their contents are

Box 1: Two silver balls

Box 2: One silver ball and one gold ball

Box 3: Two gold balls.

Carolyn picks one of the boxes at random and then picks a ball from that box at random. If that ball is gold, what is the probability that the other ball in that box is gold?

Let B_i denote the event that Carolyn picks box i (i.e., B_1 is the event that Carolyn picks Box 1, etc.). Let G denote the event that Carolyn picks a gold ball. From the problem statement, we know that

- $P(B_1) = P(B_2) = P(B_3) = 1/3$
- $P(G|B_1) = 0$
- $P(G|B_2) = 1/2$
- $P(G|B_3) = 2/3$

We are asked to find $P(B_3|G)$. We will use Bayes' Theorem, which states

$$P(X|Y) \cdot P(Y) = P(Y|X) \cdot P(X).$$

Setting $X = B_3$ and $Y = G$ in Bayes' Theorem, we get

$$P(B_3|G) \cdot P(G) = P(G|B_3) \cdot P(B_3).$$

If we know the other three numbers in the formula, then we can solve for $P(B_3|G)$. We already know two of them, namely $P(G|B_3)$ and $P(B_3)$.

We get the third, $P(G)$, from the Law of Total Probability, which states that if $Y_1, \ldots, Y_k$ partition the probability space (i.e., if exactly one of the events $Y_1, \ldots, Y_k$ is guaranteed to occur), then

$$P(X) = P(X|Y_1) \cdot P(Y_1) + \cdots + P(X|Y_k) \cdot P(Y_k).$$

Carolyn must choose exactly one of the boxes, so the events B_1, B_2, and B_3 partition the probability space. Setting $X = G$, $k = 3$, $Y_1 = B_1$, $Y_2 = B_2$, and $Y_3 = B_3$ in the Law of Total Probability we get

$$\begin{aligned} P(G) &= P(G|B_1) \cdot P(B_1) + P(G|B_2) \cdot P(B_2) + P(G|B_3) \cdot P(B_3) \\ &= 0 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} \\ &= \frac{1}{2} \end{aligned}$$

Note: There are two other ways you could have figured out $P(G)$:

- By symmetry. Swap silver and gold balls to see that $P(G) = P(\overline{G})$.
- Because all boxes have the same number of balls, picking a box at random and then a ball at random is equivalent to just picking one of the 6 balls at random so $P(G) = \frac{3}{6}$.

Keep in mind that you can't always use those two methods but you can always use the Law of Total Probability.

Now that we have our three numbers, we plug them into Bayes' Theorem:

$$\begin{aligned} P(B_3|G) \cdot P(G) &= P(G|B_3) \cdot P(B_3) \\ P(B_3|G) \cdot \frac{1}{2} &= 1 \cdot \frac{1}{3} \end{aligned}$$

so

$$P(B_3|G) = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

In other words, if the first ball is gold, the probability that the other ball is gold is $\frac{2}{3}$.

Note: In case you don't believe it, we'll derive the same answer using symmetry. Let E denote the event that both balls have the same color. This problem asked us to find $P(E|G)$. By the Law of Total Probability,

$$\begin{aligned} P(E) &= P(E|G) \cdot P(G) + P(E|\overline{G}) \cdot P(\overline{G}) \\ &= P(E|G) \cdot \frac{1}{2} + P(E|\overline{G}) \cdot \frac{1}{2} \\ &= \frac{1}{2}(P(E|G) + P(E|\overline{G})) \\ &= \frac{1}{2}(P(E|G) + P(E|G)) \qquad \text{by symmetry} \\ &= \frac{1}{2} \cdot 2 \cdot P(E|G) \qquad \text{by symmetry} \\ &= P(E|G) \qquad \text{by symmetry.} \end{aligned}$$

But the probability that both balls have the same colors is just the probability that Carolyn chooses Box 1 or Box 3, which is $\frac{2}{3}$.

A Cookie. Bill and Carolyn have two cookie jars.

Bill's jar 10 chocolate chip cookies and 10 plain cookies.

Carolyn's jar 10 chocolate chip cookies and 20 plain cookies.

During a power outage, Bill got hungry for a cookie. Because it was dark, he chose a cookie jar at random and chose a random cookie from that jar. If he got a chocolate chip cookie, what is the probability that he got it from his own jar?

Let A denote the event that Bill chooses a chocolate chip cookie. Let B denote the event that he chooses his own jar. According to the given information

$P(B) = \frac{1}{2}$

$P(A|B) = \frac{10}{20} = \frac{1}{2}$

$P(A|\overline{B}) = \frac{10}{30} = \frac{1}{3}$.

We want to know $P(B|A)$.

By Bayes' Theorem,

$$P(B|A) \cdot P(A) = P(A|B) \cdot P(B).$$

We know $P(A|B)$ and $P(B)$. We will need to figure out $P(A)$ so we can solve for $P(B|A)$.

B and $\overline{B}$ partition the probability space (i.e., exactly one of them must occur) so, by the Law of Total Probability,

$$\begin{aligned} P(A) &= P(A|B) \cdot P(B) + P(A|\overline{B}) \cdot P(\overline{B}) \\ &= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot (1 - \frac{1}{2}) \\ &= \frac{5}{12}, \end{aligned}$$

i.e., the probability that Bill got a chocolate chip cookie is $\frac{5}{12}$. Now we can apply Bayes' Theorem:

$$\begin{aligned} P(B|A) \cdot P(A) &= P(A|B) \cdot P(B) \\ P(B|A) \cdot \frac{5}{12} &= \frac{1}{2} \cdot \frac{1}{2} \\ P(B|A) \cdot \frac{5}{12} &= \frac{1}{4} \\ P(B|A) &= \frac{1}{4} \cdot \frac{12}{5} \\ P(B|A) &= \frac{3}{5}. \end{aligned}$$

Another Cookie. The next day, Bill went to the store and replaced the cookie he had eaten the night before. That night there was another blackout, and Carolyn went downstairs to get a cookie. She was startled by a strange noise and accidentally knocked over the cookie jars, whose contents poured out together into a single pile of cookies. She grabbed one cookie and ran upstairs.

What is the probability that Carolyn grabbed a chocolate chip cookie?

There were $10 + 10 = 20$ chocolate chip cookies in the pile and $20 + 30 = 50$ total cookies in the pile, so the probability that Carolyn got a chocolate chip cookie is $\frac{20}{50} = \frac{2}{5}$.

Note: This is different from the probability that Bill got a chocolate chip cookie the night before, because one jar contained more cookies than the other.

A Crookie. Shortly afterward, a burglar entered Bill and Carolyn's bedroom. Bill and Carolyn quickly reached for their gold and silver balls. The probability that Bill hits a burglar in the dark with a thrown metal ball is 10%. The probability that Carolyn hits a burglar in the dark with a thrown metal ball is 20%, independent of whether Bill hits him.

Each of them threw one metal ball at the burglar simultaneously, which scared him away. The burglar was later apprehended at a nearby hospital, where doctors determined that he had been hit by a single metal ball.

What is the probability that Bill is the one who hit him?

Let B denote the event that Bill hits the burglar. Let C denote the event that Carolyn hits the burglar. Let E denote the event that exactly one of them hits the burglar. We need to find $P(B|E)$. We know

- $P(B) = 0.1$, so $P(\overline{B}) = 0.9$
- $P(C) = 0.2$, so $P(\overline{C}) = 0.8$
- B and C are independent events

Bayes' Theorem says that

$$P(B|E) \cdot P(E) = P(E|B) \cdot P(B).$$

We already know $P(B)$. We still need to calculate $P(E)$ and $P(E|B)$ in order to solve for $P(B|E)$.

$$\begin{aligned} P(E|B) &= P(\overline{C}|B) \qquad \text{because } E \cap B = \overline{C} \cap B \\ &= P(\overline{C}) \qquad \text{by independence} \\ &= 0.8. \end{aligned}$$

$E = (B \cap \overline{C}) \cup (\overline{B} \cap C)$. That's a disjoint union, so

$$\begin{aligned} E &= B \cap \overline{C} \cup \overline{B} \cap C. \\ P(E) &= P(B \cap \overline{C}) + P(\overline{B} \cap C) \qquad \text{disjoint union} \\ &= P(B) \cdot P(\overline{C}) + P(\overline{B}) \cdot P(C) \qquad \text{independence} \\ &= 0.1 \cdot 0.8 + 0.9 \cdot 0.2 \\ &= 0.26 \end{aligned}$$

By Bayes' Theorem

$$\begin{aligned} P(B|E) \cdot P(E) &= P(E|B) \cdot P(B), \\ P(B|E) \cdot 0.26 &= 0.8 \cdot 0.1, \\ P(B|E) &= \frac{0.8 \cdot 0.1}{0.26} \\ &= \frac{4}{13} \end{aligned}$$