Math C067 Independence and Repeated Trials.

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Socks. Bill's sock drawer contains 10 red socks, 20 green socks, and 30 blue socks. If Bill removes two socks from his drawer at random, what is the probability that they have the same color?

Let E denote the event that both socks have the same color. Let R, G, and B denote the event that the first sock bill picks is red/green/blue respectively. We know:

- $P(R) = 10/60 = 1/6$
- $P(G) = 20/60 = 1/3$
- $P(B) = 30/60 = 1/2$
- $P(E|R) = 9/59$
- $P(E|G) = 19/59$
- $P(E|B) = 29/59$

Note: the two sock choices are **not** independent.

R, G, and B partition the probability space. So, by the law of total probability,

$$\begin{aligned} P(E) &= P(E|R) \cdot P(R) + P(E|G) \cdot P(G) + P(E|B) \cdot P(B) \\ &= \frac{9}{59} \cdot \frac{1}{6} + \frac{19}{59} \cdot \frac{1}{3} + \frac{29}{59} \cdot \frac{1}{2} \\ &\approx 0.378531073 \end{aligned}$$

Dice. Bill owns a pair of oddly colored 6-sided dice. Each die has 1 red face, 2 green faces, and 3 blue faces. Bill rolls both dice. What is the probability that they both come up the same color?

Let E denote the event that both dice come up the same color. Let R, G, B denote the event that that first die comes up red/green/blue respectively. We know

- $P(R) = 1/6$
- $P(G) = 2/6 = 1/3$
- $P(B) = 3/6 = 1/2$
- the results of the two die rolls are independent

R, G, and B partition the probability space. So, by the law of total probability,

$$\begin{aligned} P(E) &= P(E|R) \cdot P(R) + P(E|G) \cdot P(G) + P(E|B) \cdot P(B) \\ &= \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{2} \\ &\approx 388888889 \end{aligned}$$

When dealing with repeated **independent** trials it is conventional to write the results of the trials in order without punctionation, e.g.,

- RR denotes the event that both dice come up red
- RB denotes the event that the first die comes up red and the second blue
- BR denotes the event that the first die comes up blue and the second red

Using this notation, we would write

$$\begin{aligned} E &= RR \cup GG \cup BB \\ P(E) &= P(RR) + P(GG) + P(BB) \qquad \text{disjoint events} \\ P(E) &= \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{2} \\ P(E) &\approx 388888889 \end{aligned}$$

Three Dice. Bill rolls his colored dice 3 times. What is the probability that all three dice are the same color? What is the probability that the three dice are all different colors?

Let E denote the event that all three dice are the same color.

$$\begin{aligned}
E &= RRR \cup GGG \cup BBB \\
P(E) &= P(RRR) + P(GGG) + P(BBB) \qquad \text{disjoint events} \\
P(E) &= \frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6} + \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} \\
P(E) &\approx 0.166666667
\end{aligned}$$

Let E denote the event that the three dice are all different colors.

$$\begin{aligned}
E &= RGB \cup RBG \cup BRG \cup BGR \cup GRB \cup GBR \\
P(E) &= P(RGB) + P(RBG) + P(BRG) + P(BGR) + P(GRB) + P(GBR) \\
P(E) &= \frac{1}{6}\cdot\frac{1}{3}\cdot\frac{1}{2} + \frac{1}{6}\cdot\frac{1}{2}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{6}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{6} + \frac{1}{2}\cdot\frac{1}{6}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{6} \\
P(E) &= 6\cdot\frac{1}{6}\cdot\frac{1}{2}\cdot\frac{1}{3} \\
P(E) &= \frac{1}{2}\cdot\frac{1}{3} \\
P(E) &= \frac{1}{6}
\end{aligned}$$

Note: it's only a coincidence that the two probabilities are the same.

4.49 Let A and B be independent events with $p(A) = 0.3$ and $p(B) = 0.4$. Find $p(A \cap B)$, $p(A \cup B)$, $P(A|B)$ and $p(B|A)$.

- Because A and B are independent, $p(A \cap B) = p(A) \cdot p(B) = 0.3 \cdot 0.4 = 0.12$
- $p(A \cup B) = p(A) + p(B) - p(A \cap B) = 0.3 + 0.4 - 0.12 = 0.58$.
- Because A and B are independent, $p(A|B) = p(A) = 0.3$.
- Because A and B are independent, $p(B|A) = p(B) = 0.4$.

4.50 Box A contains 5 red marbles and 3 blue marbles. Box B contains 2 red and 3 blue. A marble is drawn at random from each box.

(a) Find the probability that both marbles are red.

(b) Find the probability that both marbles are red.

Solution: Let R_1 denote the event that the first marble is red, B_1 the event that the first marble is blue, R_2 denote the event that the second marble is red, B_2 the event that the second marble is blue.

(a) Because the two marbles are chosen randomly from different boxes, their choices are independent.

$$\begin{aligned} p(R_1 \cap R_2) &= p(R_1) \cdot p(R_2) \qquad \text{independence} \\ &= \frac{5}{8} \cdot \frac{2}{5} \\ &= \frac{2}{8} \\ &= \frac{1}{4} \end{aligned}$$

(b)

$$\begin{aligned} p(R_1 \cap B_2 \cup B_1 \cap R_2) &= p(R_1 \cap B_2) + p(B_1 \cap R_2) \qquad \text{disjoint events} \\ &= p(R_1) \cdot p(B_2) + p(B_1) \cdot p(R_2) \qquad \text{independence} \\ &= \frac{5}{8} \cdot \frac{3}{5} + \frac{3}{8} \cdot \frac{2}{5} \\ &= \frac{3}{8} + \frac{3}{20} \\ &= \frac{21}{40} \end{aligned}$$

4.51 Let A and B be events with $p(A) = 0.3$, $p(A \cup B) = 0.5$, and $p(B) = p$. Find p if:

(a) A and B are disjoint

(b) A and B are independent

(c) A is a subset of B

Solution: $p(A)$, $p(B)$, and $p(A \cup B)$ are always related via the formula

$$p(A \cup B) = p(A) + p(B) - p(A \cap B).$$

If we know the other three numbers in the formula, we can solve for $p(B)$. We are given $p(A)$ and $p(A \cup B)$, so it remains for us to figure out $p(A \cap B)$.

(a) We are told that A and B are disjoint, i.e., that $A \cap B = \emptyset$, so $p(A \cap B) = 0$.

$$\begin{aligned} p(A \cup B) &= p(A) + p(B) - p(A \cap B) \\ 0.5 &= 0.3 + p - 0 \\ 0.2 &= p \end{aligned}$$

Alternatively, we could have started this part with the equation

$$\begin{aligned} p(A \cup B) &= p(A) + p(B) \qquad \text{because } A \text{ and } B \text{ are disjoint} \\ 0.5 &= 0.3 + p \\ 0.2 &= p \end{aligned}$$

(b) We are told that A and B are independent, so $p(A \cap B) = P(A) \cdot p(B)$.

$$\begin{aligned} p(A \cup B) &= p(A) + p(B) - p(A \cap B) \\ 0.5 &= 0.3 + p - 0.3 \cdot p \\ 0.2 &= 0.7p \\ \frac{0.2}{0.7} &= p \\ \frac{2}{7} &= p \end{aligned}$$

(c) We are told that A is a subset of B, so $A \cap B = A$, so $p(A \cap B) = p(A)$.

$$\begin{aligned} p(A \cup B) &= p(A) + p(B) - p(A \cap B) \\ 0.5 &= 0.3 + p - 0.3 \\ 0.5 &= p \end{aligned}$$

Alternatively, we could use the formula

$$\begin{aligned} A \cup B &= B \qquad \text{because } A \text{ is a subset of } B \\ p(A \cup B) &= p(B) \\ 0.5 &= p \end{aligned}$$

1. Important General Formulas for Computing Ordinary Probabilities.

Union (OR) $p(A \cup B) = p(A) + p(B) - p(A \cap B)$

Intersection (AND) $p(A \cap B) = p(A|B) \cdot p(B)$

Complementation (NOT) $p(\overline{A}) = 1 - p(A)$

Law of Total Probability $p(A) = p(A|B) \cdot p(B) + p(A|\overline{B}) \cdot p(\overline{B})$

2. Important Formulas for Independent Events A and B

Conditional Probability (independent events)

- $p(A|B) = p(A)$
- $p(A|\overline{B}) = p(A)$
- $p(\overline{A}|B) = p(\overline{A})$
- $p(\overline{A}|\overline{B}) = p(\overline{A})$

Intersection (independent events) $p(A \cap B) = p(A) \cdot p(B)$

Union (independent events) First apply the general formula for union, and then apply the intersection formula for independent events. You will get

$$p(A \cup B) = p(A) + p(B) - p(A) \cdot p(B)$$

but there is no need to memorize this is a separate formula.

3. Important Formulas for Computing Conditional Probabilities

$p(A|B) = p(A \cap B|B)$

$p(A|B) = |A \cap B|/|B|$

$p(A|B) = p(A \cap B)/p(B)$

Bayes' Theorem $p(A|B) \cdot p(B) = p(B|A) \cdot p(A)$