Math C067 — Example

Richard Beigel

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1. A Two-Fund Portfolio

- Stocks X and Y are the same as in the first week's Homework assignment
- From homework: A mutual fund that invests 1/3 in Stock X and 2/3 in Stock Y is still riskier than stock Y.
- What is the right way to weight X and Y when designing a mutual fund?
- Suggestions, please.

2. A Two-Fund Portfolio

- Stocks X and Y are the same as in the first week's Homework assignment
- From homework: A mutual fund that invests 1/3 in Stock X and 2/3 in Stock Y is still riskier than stock Y.
- What is the right way to weight X and Y when designing a mutual fund?
- What if we weight X and Y equally? Would this be better or worse than $\frac{1}{3}$:$\frac{2}{3}$?
- Should we weight X more heavily than Y, or weight Y more heavily than X?

3. A Two-Fund Portfolio

- Stocks X and Y are the same as in the first week's Homework assignment
- Mutual Fund Z invests 13% in stock X and 87% in stock Y
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Year	1	2	3	4
Return on Stock X	0	0	8	0
Return on Stock Y	-1	4	3	2
Return on Stock Z	-0.87	3.48	3.65	1.74

- $\bar{z} = 0.13\bar{x} + 0.87\bar{y} = 0.13 \cdot 2 + 0.87 \cdot 2 = 2$
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$$\begin{aligned} s^2 &= \left((-0.87-2)^2 + (3.48-2)^2 + (3.65-2)^2 + (1.74-2)^2\right)/3 \\ &= \left((-2.87)^2 + 1.48^2 + 1.65^2 + (-0.26)^2\right)/3 \\ &\approx 13.2174/3 \\ &\approx 4.4 \end{aligned}$$

- Recall that the variance for Stock X is 16 and the variance for Stock Y is approximately 4.67. Since X, Y, and Z have the same expected return (mean) and Z has the lowest risk (variance), Z is preferable.

4. Where did 13% come from? Minimizing Risk (theory)

We will figure out how to weight two stocks optimally, assuming they the same expected return (mean), in order to minimize the variance of the resulting portfolio.

- Portfolio Z will give weight w to Stock X and weight $1 - w$ to stock Y.
- i.e., $z_i = wx_i + (1 - w)y_i$ for each i
- Z is called a weighted average of X and Y
- Easy formula for the mean of Z:

$$\bar{z} = w\bar{x} + (1 - w)\bar{y}$$

- In our example $\bar{x} = \bar{y}$, so $\bar{z} = w\bar{x} + (1 - w)\bar{x} = \bar{x}$
- Not-so-easy formula for the variance of Z:

$$s_z^2 = w^2 s_x^2 + 2w(1 - w)s_{xy} + (1 - w)^2 s_y^2$$

- Plug in values for s_x^2, s_{xy}, and s_y^2
- Simplify using the distributive law
- The result is a quadratic in the variable w:

$$s_z^2 = aw^2 - bw + c$$

  a, b, and c are numbers that depend on s_x^2, s_y^2, and s_{xy}.

- Formula for minimizing a quadratic:

$$w = \frac{b}{2a}$$

  (Divide the coefficient of the degree-1 term by twice the coefficient of the degree-2 term, and change the sign.)

- *These are not the same a and b as from the best-fit line.*

5. Minimizing Risk (using the formulas)

- Stocks X and Y are as above.
- Mutual Fund Z invests a fraction w of its money in stock X, and $(1-w)$ in stock Y.
- Recall that $s_x^2 = 16$, $s_{xy} = 8/3$, and $s_y^2 = 14/3$.

$$\begin{aligned}
s_z^2 &= w^2 s_x^2 + 2w(1-w)s_{xy} + (1-w)^2 s_y^2 \\
s_z^2 &= w^2 \cdot 16 + 2 \cdot w(1-w) \cdot \frac{8}{3} + (1-w)^2 \cdot \frac{14}{3} \\
s_z^2 &= 16w^2 + 2 \cdot \frac{8}{3} w(1-w) + \frac{14}{3}(1-w)^2 \\
s_z^2 &= 16w^2 + \frac{16}{3}(w - w^2) + \frac{14}{3}(1 - 2w + w^2) \\
s_z^2 &= 16w^2 + \frac{16}{3}w - \frac{16}{3}w^2 + \frac{14}{3} \cdot 1 - \frac{14}{3} \cdot 2 \cdot w + \frac{14}{3}w^2 \\
s_z^2 &= 16w^2 + \frac{16}{3}w - \frac{16}{3}w^2 + \frac{14}{3} - \frac{28}{3}w + \frac{14}{3}w^2 \\
s_z^2 &= 16w^2 - \frac{16}{3}w^2 + \frac{14}{3}w^2 + \frac{16}{3}w - \frac{28}{3}w + \frac{14}{3} \\
s_z^2 &= (16 - \frac{16}{3} + \frac{14}{3})w^2 + (\frac{16}{3} - \frac{28}{3})w + \frac{14}{3} \\
s_z^2 &= \frac{46}{3}w^2 - 4w + \frac{14}{3}
\end{aligned}$$

- The quadratic is minimized by taking $w = 4/(2 \cdot 46/3) = 3/23 \approx 0.130434783$.
- The resulting variance is

$$\frac{46}{3} \cdot (3/23)^2 - 4(3/23) + 14/3 = 304/69 \approx 4.4057971$$