Math C067 Binomial and Normal Distributions

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1. Bernoulli Trials Consider an experiment with two possible outcomes, called **success** and **failure**. Each outcome has a specified probability: p for success and q for failure (so that $p + q = 1$).

- $P(\text{success}) = p$
- $P(\text{failure}) = q = 1 - p$

If such an experiment is repeated several times **independently**, the experiments are called **Bernoulli trials**.

2. Binomial Distribution Let X denote the number of successes when we perform n Bernouilli trials.

The distribution of X is called a **Binomial distribution** and is denoted $\boldsymbol{B(n, p)}$.

- $P(X = k) = \binom{n}{k} p^k q^{n-k}$

  Note: k is the number of successes and $n - k$ is the number of failures
- Special case: $P(X = 0) = q^n$
- Special case: $P(X = n) = p^n$
- $P(X \geq k) = P(X = k) + P(X = k + 1) + \cdots + P(X = n)$
- Special case: $P(X \geq 1) = 1 - P(X = 0) = 1 - q^n$

How do we calculate $\binom{n}{k}$?

Method 1: Use the formulas in Chapter 2. Useful special cases:

- $\binom{n}{0} = \binom{n}{n} = 1$
- $\binom{n}{1} = \binom{n}{n-1} = n$

Method 2: Points in the sample space are n-tuples of successes and failures, which can be represented as sequences of n S's and F's. Make a list of all the points in the sample space. The number of points with exactly k S's (and $n-k$ F's) is $\binom{n}{k}$.

Method 3: Make a list of all sequences consisting of exactly k S's and $n-k$ F's. The number of such sequences is $\binom{n}{k}$.

Example: Betting on heads. Say we toss a coin 3 times and bet on heads each time. Let X denote the number of times that we win.

- $p = P(\text{success}) = \frac{1}{2}$
- $q = 1 - p = \frac{1}{2}$
- X is binomially distributed with distribution $B(3, \frac{1}{2})$.
- Sample space = {HHH,HHT,HTH,HTT,THH,THT,TTH,TTT}.
- There is one point with 0 heads, so $\binom{3}{0} = 1$
- There are 3 points with 1 heads, so $\binom{3}{1} = 3$
- There are 3 points with 2 heads, so $\binom{3}{2} = 3$
- There is one point with 3 heads, so $\binom{3}{3} = 1$
- $P(X = 0) = \binom{3}{0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^3 = 1 \cdot 1 \cdot \frac{1}{8} = \frac{1}{8}$
- $P(X = 1) = \binom{3}{1} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^2 = 3 \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{3}{8}$
- $P(X = 2) = \binom{3}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1 = 3 \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{8}$
- $P(X = 3) = \binom{3}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^0 = 1 \cdot \frac{1}{8} \cdot 1 = \frac{1}{8}$
- $P(X \geq 0) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$
- $P(X \geq 1) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$
- $P(X \geq 2) = \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$
- $P(X \geq 3) = \frac{1}{8}$

k	0	1	2	3
$P(X = k)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
$P(X \geq k)$	1	$\frac{7}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

Example: Spinning a Biased Coin. Same experiment, but the coin is biased so that it comes up heads $\frac{2}{3}$ of the time. Let X denote the number of times that we win.

- $p = P(\text{success}) = \frac{2}{3}$
- $q = 1 - p = \frac{1}{3}$
- X is binomially distributed with distribution $B(3, \frac{2}{3})$.
- Sample space = {HHH,HHT,HTH,HTT,THH,THT,TTH,TTT} (not equiprobable).
- There is one point with 0 heads, so $\binom{3}{0} = 1$
- There are 3 points with 1 heads, so $\binom{3}{1} = 3$
- There are 3 points with 2 heads, so $\binom{3}{2} = 3$
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- $P(X = 1) = \binom{3}{1}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^2 = 3 \cdot \frac{2}{3} \cdot \frac{1}{9} = \frac{2}{9}$
- $P(X = 2) = \binom{3}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^1 = 3 \cdot \frac{4}{9} \cdot \frac{1}{3} = \frac{4}{9}$
- $P(X = 3) = \binom{3}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^0 = 1 \cdot \frac{8}{27} \cdot 1 = \frac{8}{27}$
- $P(X \geq 0) = \frac{1}{27} + \frac{2}{9} + \frac{4}{9} + \frac{8}{27} = 1$
- $P(X \geq 1) = \frac{2}{9} + \frac{4}{9} + \frac{8}{27} = \frac{26}{27}$
- $P(X \geq 2) = \frac{4}{9} + \frac{8}{27} = \frac{20}{27}$
- $P(X \geq 3) = \frac{8}{27}$

k	0	1	2	3
$P(X = k)$	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$
$P(X \geq k)$	1	$\frac{26}{27}$	$\frac{20}{27}$	$\frac{8}{27}$

Example: Rolling a die Say we roll a 6-sided die 3 times. Each time, we bet that the die will come up 1, 2, 3, or 4. Let X denote the number of times that we win.

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What's our sample space?

The equiprobable sample space, consisting of all 3-tuples of numbers between 1 and 6, is big. It has $6^3 = 216$ points.

It would be nice to work with a smaller sample space.

Example: It's not how you play the game; It's whether you win or lose Say we roll a 6-sided die 3 times. Each time, we bet that the die will come up 1, 2, 3, or 4. Let X denote the number of times that we win.

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Idea: the actual number on the die isn't very important. What matters is whether the trial is a success or a failure.

Example: Rolling a die Say we roll a 6-sided die 3 times. Each time, we bet that the die will come up 1, 2, 3, or 4. Let X denote the number of times that we win.

- $p = P(\text{success}) = \frac{4}{6} = \frac{2}{3}$
- $q = 1 - p = \frac{1}{3}$
- X is binomially distributed with distribution $B(3, \frac{2}{3})$.
- Sample space = {SSS,SSF,SFS,SFF,FSS,FSF,FFS,FFF} (not equiprobable).
- There is one point with 0 successes, so $\binom{3}{0} = 1$
- There are 3 points with 1 successes, so $\binom{3}{1} = 3$
- There are 3 points with 2 successes, so $\binom{3}{2} = 3$
- There is one point with 3 successes, so $\binom{3}{3} = 1$
- $P(X = 0) = \binom{3}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^3 = 1 \cdot 1 \cdot \frac{1}{27} = \frac{1}{27}$
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- $P(X = 3) = \binom{3}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^0 = 1 \cdot \frac{8}{27} \cdot 1 = \frac{8}{27}$
- $P(X \geq 0) = \frac{1}{27} + \frac{2}{9} + \frac{4}{9} + \frac{8}{27} = 1$
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- $P(X \geq 2) = \frac{4}{9} + \frac{8}{27} = \frac{20}{27}$
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k	0	1	2	3
$P(X = k)$	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$
$P(X \geq k)$	1	$\frac{26}{27}$	$\frac{20}{27}$	$\frac{8}{27}$

Example: A single "Bernoulli" trial. Let X be the number of successes when we perform a single "Bernoulli" trial. The mean and variance of a random variable X with distribution $B(1, p)$ are calculated below: We have

- $P(X = 1) = p$ (one success)
- $P(X = 0) = q = 1 - p$ (zero successes $\equiv$ one failure)
- $E(X) = 0 \cdot P(X = 0) + 1 \cdot P(X = 1) = P(X = 1) = p$
-

$$\begin{aligned}
\text{Var}(X) &= E(X^2) - (E(X))^2 \\
&= E(X^2) - p^2 \qquad \text{Why? Because } E(X) = p \\
&= E(X) - p^2 \qquad \text{Why? } X^2 = X \text{ because } R_X = \{0, 1\} \\
&= p - p^2 \\
&= p(1 - p) \\
&= pq
\end{aligned}$$

Mean, Variance, and Standard Deviation for $\boldsymbol{B(n, p)}$ The mean, variance, and standard deviation of a random variable X with distribution $B(n, p)$ are given by the following formulas:

Expected value $\mu = n \cdot p$ (even if the trials weren't independent)

Variance $\sigma^2 = n \cdot pq$ (because the trials are independent)

Standard Deviation $\sigma = \sqrt{npq}$

Example: Flipping a Fair Coin Say we flip a fair coin 100 times. Let X denote the number of heads that we get. Then X has distribution $B(100, \frac{1}{2})$.

- $E[X] = 100 \cdot \frac{1}{2} = 50$
- $\text{Var}[X] = 100 \cdot \frac{1}{2} \cdot \frac{1}{2} = 25$
- $\sigma_X = \sqrt{\text{Var}[x]} = \sqrt{25} = 5.$

Example: Spinning a Biased Coin Say we flip a biased coin 100 times. Assume that the coin has probability $\frac{2}{3}$ of coming up heads. Let X denote the number of heads that we get. Then X has distribution $B(100, \frac{2}{3})$.

- $E[X] = 100 \cdot \frac{2}{3} = \frac{200}{3} \approx 67$
- $\text{Var}[X] = 100 \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{200}{9} \approx 22.2$
- $\sigma_X = \sqrt{\text{Var}[x]} = \sqrt{\frac{200}{9}} \approx 4.71$

Example: Rolling a Die Say we roll a die 100 times. Each time we bet that the die will come up 1, 2, 3, or 4. Let X denote the number of times we win our bet. Then X has distribution $B(100, \frac{2}{3})$.

- $E[X] = 100 \cdot \frac{2}{3} = \frac{200}{3} \approx 167$
- $\text{Var}[X] = 100 \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{200}{9} \approx 22.2$
- $\sigma_X = \sqrt{\text{Var}[x]} = \sqrt{\frac{200}{9}} \approx 4.71$

Using Table A-1 on page 359

Example: Flipping a Fair Coin Say we flip a fair coin 100 times.

1. What is the (approximate) probability that we get at least 45 heads but at most 60 heads?

Solution: Let X denote the number of heads that we get. Then X has distribution $B(100, \frac{1}{2})$.

1. Make sure $np \geq 5$ and $nq \geq 5$.
2. Calculate mean: $\mu = 100 \cdot \frac{1}{2} = 50$
3. Calculate variance: $\sigma^2 = 100 \cdot \frac{1}{2} \cdot \frac{1}{2} = 25$
4. Calculate standard deviation: $\sigma = \sqrt{\mathrm{Var}[x]} = \sqrt{25} = 5$.
5. What do we want to estimate? $P(45 \leq X < 60)$.
6. Split the intervals between possible values: $P(45 \leq X < 60) = P(44.5 \leq X \leq 59.5)$
7. Split into left of mean and right of mean:
$$P(44.5 \leq X \leq 59.5) = P(44.5 \leq X \leq 50) + P(50 < X \leq 59.5)$$
8. How many standard deviations on each side of the mean?
   (a) $(50 - 44.5)/5 = 1.1$ on the left, so $P(44.5 \leq X \leq 50) \approx 0.3643$ (from Table A-1)
   (b) $(59.5 - 50)/5 = 1.9$ on the right, so $P(50 < X \leq 59.5) \approx 0.4713$ (from Table A-1)
9. Add:
$$P(44.5 \leq X \leq 59.5) = P(44.5 \leq X \leq 50) + P(50 < X \leq 59.5) \approx 0.3643 + 0.4713 = 0.8356$$

What is the (approximate) probability that we get exactly 50 heads?

Solution:

1. Make sure $np \geq 5$ and $nq \geq 5$.
2. What do we want to estimate? $P(50 \leq X \leq 50)$.
3. Split the intervals between possible values: $P(50 \leq X \leq 50) = P(49.5 \leq X \leq 50.5)$
4. Split into left of mean and right of mean:
$$P(49.5 \leq X \leq 50.5) = P(49.5 \leq X \leq 50) + P(50 < X \leq 50.5)$$
5. How many standard deviations on each side of the mean?
   (a) $(50 - 49.5)/5 = 0.1$ on the left, so $P(49.5 \leq X \leq 50) \approx 0.0398$ (from Table A-1)
   (b) $(50.5 - 50)/5 = 0.1$ on the right, so $P(50 < X \leq 50.5) \approx 0.0398$ (from Table A-1)
6. Add:
$P(49.5 \leq X \leq 50.5) = P(50 \leq X \leq 50.5) + P(50 < X \leq 50.5) \approx 0.0398 + 0.0398 = 0.0796$

What is the (approximate) probability that we get at least 65 heads?

Solution:

1. Make sure $np \geq 5$ and $nq \geq 5$.
2. What do we want to estimate? $P(65 \leq X)$.
3. Split the intervals between possible values: $P(65 \leq X) = P(64.5 \leq X)$
4. Write as a complement:
$$P(64.5 \leq X) = P(50 \leq X) - P(50 \leq X < 64.5)$$
5. How many standard deviations?
   (a) $(64.5 - 50)/5 = 2.9$
6. Subtract:
$$P(64.5 \leq X) = P(50 \leq X) - P(50 \leq X < 64.5) \approx 0.5 - 0.4981 = 0.0019$$
Why do we subtract? Because we are really calculating $P(64.5 \leq X \leq 100)$, and the numbers 64.5 and 100 are both on the same side of the mean.
   - We **add** when the endpoints are on **opposite** sides of the mean.
   - We **subtract** when both endpoints are on the **same** side of the mean.

   See the third picture in Fig. 6-4 on page 185. See also Example 6.25(b) on pp. 199-200. The general idea is explained in Section 6.4. I highly recommend that you read Sections 6.1-6.5.

Exercises:

1. A biased coin is spun 3 times. The probability that it comes up heads is $\frac{1}{5}$. What is the probability that we get
   (a) exactly one heads?
   (b) at least one heads?
   (c) three heads?

2. A biased coin is spun 100 times. The probability that it comes up heads is $\frac{4}{5}$. Let X denote the number of times the coin comes up heads. Let Y denote the number of times the coin comes up tails. What is
   (a) the expected value of X?
   (b) the variance of X?
   (c) the standard deviation of X?
   (d) the expected value of Y?
   (e) the variance of Y?
   (f) the standard deviation of Y?

3. A biased coin is spun 100 times. The probability that it comes up heads is $\frac{4}{5}$. Let X denote the number of times the coin comes up heads. Estimate:
   (a) $P(X = 80)$
   (b) $P(65 \leq X \leq 75)$
   (c) $P(70 \leq X \leq 80)$
   (d) $P(75 \leq X \leq 85)$
   (e) $P(80 \leq X \leq 90)$
   (f) $P(85 \leq X \leq 95)$
   (g) $P(X \geq 75)$
   (h) $P(X \geq 80)$
   (i) $P(X \geq 85)$
   (j) $P(X \leq 75)$
   (k) $P(X \leq 80)$
   (l) $P(X \leq 85)$

4. A die is rolled 45 times. Let X denote the number of times that it comes up 6. What is
   (a) the expected value of X?
   (b) the variance of X?
   (c) the standard deviation of X?

5. A die is rolled 45 times. Let X denote the number of times that it comes up 6. Estimate

(a) $P(X = 7.5)$

(b) $P(5 \leq X \leq 10)$

(c) $P(10 \leq X \leq 20)$

(d) $P(X \leq 5)$

(e) $P(X \leq 8)$

(f) $P(X \leq 15)$

(g) $P(X \geq 5)$

(h) $P(X \geq 8)$

(i) $P(X \geq 15)$