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WELL-COMPOSED SETS

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Abstract

A special class of subsets of binary digital images called "well-composed sets" is defined. The sets of this class have very nice topological properties; for example, the Jordan Curve Theorem holds for them, their Euler characteristic is locally computable, and they have only one connectedness relation, since 4- and 8-connectedness are equivalent. This implies that many basic algorithms used in computer vision become simpler.

There are real advantages in applying thinning algorithms to well-composed sets. For example, thinning is an internal operation on these sets and the problems with irreducible "thick" sets disappear. Furthermore, we prove that the skeletons obtained are "one point thick" and we give a formal definition of this concept. We also show that these skeletons have a graph structure and we define what this means.

List of symbols:

$\emptyset$ – the empty set

$\subset$ – set inclusion

$\cap$ – intersection

$\cup$ – union

$\in$ – element of

$\notin$ – not element of

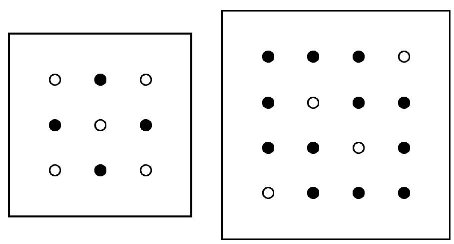
$\mathbb{Z}$ – integers

$\setminus$ – set difference

$\overline{}$ – set complement

1 Introduction

The most often used representation space in digital image processing is a rectangular array, which can be regarded as a finite subset of $\mathbb{Z}^2$. In the case of binary digital images, two different sorts of points are treated: black points for the foreground and white points for the background. In $\mathbb{Z}^2$, two adjacency relations can be defined: 4-adjacency and 8-adjacency. As was noted early in the history of computer vision, using the same adjacency relation for all points of a digital picture leads to so-called "paradoxes" such as those pointed out in [18] (see also [10]). We illustrate these paradoxes by means of the two sets of black points in Figure 1. If 4-adjacency is used for all pairs of points in Figure 1a, then the black points are totally disconnected. However, they separate the set of white points into two components. If 8-adjacency is used for all pairs of points, then the black points form a discrete analog of a Jordan curve (simple closed curve), but they do not separate the white points.



(a) (b)

Figure 1: Connectivity paradoxes

Thus a digital version of the Jordan Curve Theorem does not hold for 8-adjacency (the Jordan Curve Theorem states that the complement of a simple closed curve in the plane always has exactly two components, which can be interpreted as the interior and the exterior of the curve). Figure 1b shows that the situation is no better for 4-adjacency. The black points constitute a 4-connected simple closed curve, but there exist three 4-connected components of the background. Thus in either case, a digital version of the Jordan Curve Theorem does not hold.

The most popular solution to these problems was the idea of using different adjacency relations for the foreground and the background: 8-adjacency for black points and 4-adjacency for white points, or vice versa (first recommended in [5]). Rosenfeld [15] developed the foundations of digital topology based on this idea, and showed that the Jordan curve theorem then holds.

The price we have to pay for this solution is that we have two different adjacency relations in one digital picture which depend on the objects being represented. Therefore, the adjacency relation is not an intrinsic feature of a digital picture as a representing medium. Since we have one connectedness relation for the foreground and one for the background, by interchanging the foreground and the background, we also change the connectedness relations of the digital picture. Consequently, the choice of foreground and of background can be critical, especially in cases in which it is not clear what is the foreground and what is the background, because this choice immediately determines the connectedness structure of the digital picture.

In this paper, we present another solution which allows us to avoid the connectivity paradoxes while having only one connectedness relation for the entire digital picture. We will use only 4-connectedness (which will be equivalent to 8-connectedness) for black as well as for white points. The idea is not to treat all decompositions of $\mathbb{Z}^2$ into foreground and background as digital pictures, but only a special class which define what we call "well-composed sets". [This idea has a real mathematical flavor, since in most fields of mathematics we do not treat all subsets of a given space, but only a class of subsets which have "nice properties" with regard to features we are especially interested in.] Specifically, we will define a digital picture to be well-composed if every 8-component of either the foreground or the background is 4-connected (and hence is a 4-component).

Requiring digital pictures to be well-composed is actually a consequence of requiring that the process of digitization preserve topology. In this paper we define digitization using a grid of squares, say of diameter d (see Section 3); a square is a black pixel iff it contains a black point. As Figure 2 shows, a disconnected set S of black pixels can have a connected digitization. Note that in the ball of radius d, the intersection of S with two diagonally adjacent squares is disconnected by the other two diagonally adjacent squares which do not intersect S. If this situation occurs, the digitization process (to an 8/4 picture) will not be topology preserving, as the figure illustrates. So, if the digitization process of a set S is topology preserving, such a situation cannot occur. In this case, the resulting picture will always be well-composed, since it will not contain four corner-adjacent squares such that two diagonally adjacent squares are black and the other two are white. Thus well-composedness is guaranteed if digitization preserves topology.

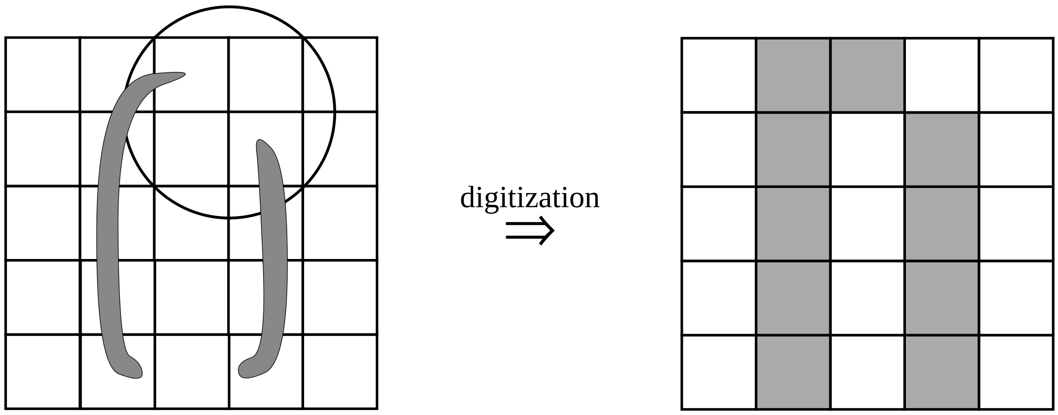


Figure 2: A disconnected set can have a connected digitization.

An obvious advantage of treating only well-composed digital pictures is that their digital topological description is much simpler. Since we need only one adjacency relation, namely 4-adjacency, we do not have to decide which adjacency relation to use for a given pair of points. If we use two different adjacency relations, then whichever one we use for the foreground, we must use the other one for the background, which complicates the digital topological description. The advantages of well-composedness as regards the Jordan Curve Theorem and the Euler characteristic are discussed in Section 4.

As we will see in Sections 5 and 6, there are great advantages in applying thinning algorithms to well-composed sets. Thinning is an internal operation on these sets and the problems with irreducible "thick" sets disappear. We prove that simple thinning algorithms can be defined for well-composed sets which generate really "thin" skeletons which are "one point thick", and we also formally define what this means. We also show that skeletons have a graph structure and we define what this means.

Thinning algorithms are simpler on well-composed sets, since only 1/3 as many neighborhood configurations need be considered to decide if a given point can be deleted: in general, there are 18 types of 3×3 neighborhoods of simple points (other than endpoints), which generate 108 neighborhood configurations by rotations and reflections; of these, only 7 types are neighborhoods of simple points (other than endpoints) in well-composed sets, which generate only 36 configurations. Parallel thinning algorithms for well-composed sets are discussed in Section 8.

2 Basic Definitions

In this section we review some definitions from digital topology which are based on [15] and [10]. As usual in two dimensional digital topology, we assume that all sets are subsets of the *digital plane* $\mathbb{Z}^2$.

Let S be a subset of the digital plane; the points in S will be termed *black* or *foreground* points, while those of the complement $\overline{S}$ of S will be termed *white* or *background* points.

Definition 2.1 *The* 4-neighbors *(or* direct neighbors*) of a point* (x, y) *in* $\mathbb{Z}^2$ *are its four horizontal and vertical neighbors* $(x+1, y)$, $(x-1, y)$ *and* $(x, y+1)$, $(x, y-1)$. *The* 8-neighbors *of a point* (x, y) *in* $\mathbb{Z}^2$ *are its four horizontal and vertical neighbors together with its four diagonal neighbors* $(x+1, y+1)$, $(x+1, y-1)$ *and* $(x-1, y+1)$, $(x-1, y-1)$.

For $n = 4$ or 8, the *n-neighborhood* of a point $P = (x, y)$ in $\mathbb{Z}^2$ is the set $\mathcal{N}_n(P)$ consisting of P and its n-neighbors. $\mathcal{N}_n^*(P)$ is the set of all neighbors of P without P itself. Note that $\mathcal{N}_n^*(P) = \mathcal{N}_n(P) \setminus \{P\}$.

The points in $\mathcal{N}_8^*(P)$ are numbered 0 to 7 according to the following scheme:

$N_3(P)$	$N_2(P)$	$N_1(P)$
$N_4(P)$	P	$N_0(P)$
$N_5(P)$	$N_6(P)$	$N_7(P)$

Each neighborhood can be characterized by a number $B(P)$. Let $d_i = 1$ if $N_i(P) \in S$ and $d_i = 0$ if $N_i(P) \notin S$; then

$$B(P) = \sum_{i=0}^{7} d_i \cdot 2^i.$$

If a neighborhood configuration is given in the text, we sometimes write the number $B(P)$ in the center. Among configurations which are equivalent with respect to rotations or reflections of the digital plane, we usually take the one that has the smallest value of $B(P)$ as a representative.

Definition 2.2 *Let P, Q be any points of $\mathbb{Z}^2$. By a* path *from P to Q we mean a sequence of points $P = P_1, P_2, \ldots, P_n = Q$ such that P_i is a neighbor of P_{i-1}, $1 < i \leq n$.*

Depending on whether we use 4- or 8-neighborhoods, we call this path a 4- or an 8-path.

Definition 2.3 *A set X is* 4-connected (8-connected) *if for every pair of points P, Q in X, there is a 4-path (8-path) in X from P to Q.*

Sometimes we will also say that X is *connected*, which means X is 4-connected if we treat 4-paths or X is 8-connected if we treat 8-paths.

Definition 2.4 *A* component *of a set S is a greatest connected subset of S. Depending on whether 4- or 8-connectedness is used, we have 4- or 8-components.*

In particular, if S is connected, then the only component of S is S itself.

Definition 2.5 *A set C is called a* simple closed curve (Jordan curve) *if it is connected, and each of its points has exactly two neighbors in C. Depending on whether we use 4- or 8-neighborhoods, we can call C a 4- or 8-curve.*

To avoid pathological situations we require that a 4-curve contain at least 8 points and an 8-curve at least 4 points [15].

A set A is called an arc *if it is connected, and all but two of its points have exactly two neighbors in A, while these two have exactly one. These two points are called* endpoints. *Depending on whether we use 4- or 8-neighborhoods, we can call A a 4- or 8-arc.*

Definition 2.6 *A point of a digital set S having all four of its direct neighbors also in S is called an* interior point. *The set of all interior points of a set S is termed the* kernel *of S. All points in S which are not interior points are called* boundary points.

In this paper, for illustration purposes, we denote the different types of points by the following symbols:

$\bullet$	:	black point
$\circ$	:	(or blank position) white point
$\blacksquare$	:	interior point
$\cdot$	:	point of either color.

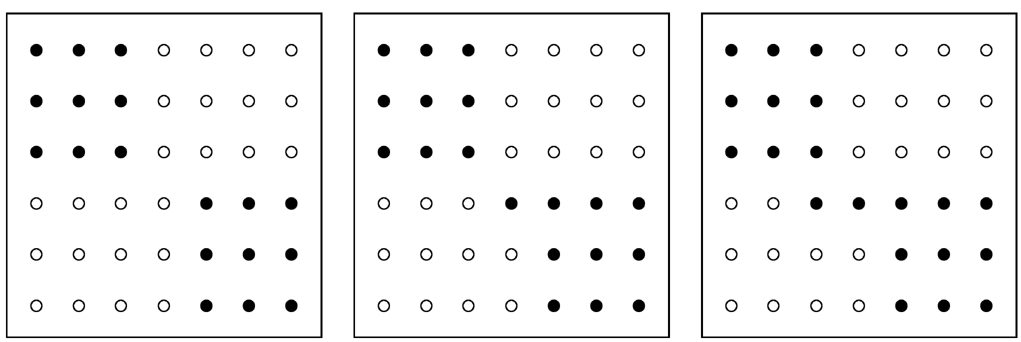
3 Well-Composedness

First we give a few basic definitions. All of the following sets are subsets of $\mathbb{Z}^2$.

Definition 3.1 *S is* weakly well-composed *if any 8-component of S is a 4-component.*

Definition 3.2 *S is* well-composed *if both S and its complement $\overline{S}$ are weakly well-composed.*

Examples are shown in Figures 3 and 4.



(a) (b) (c)

Figure 3: The sets in (a) and (c) are well-composed; the set in (b) is neither well-composed nor weakly well-composed.

Since sets like the two sets of black points in Figure 1 cannot be well-composed, the connectivity paradoxes described in the introduction simply do not occur for well-composed sets. In Section 4 we will prove the Jordan Curve Theorem for well-composed sets. The following definition gives a local characterization of well-composed sets, as we show in Theorem 3.1.

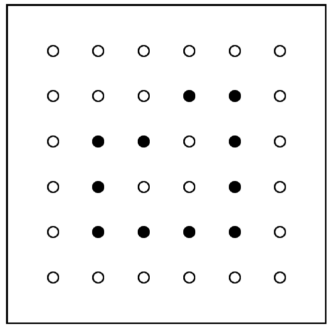


Figure 4: A weakly well-composed set which is not well-composed.

Definition 3.3 *S is* locally 4-connected *if the points of S in the 8-neighborhood of any point of S are 4-connected, i.e. $S \cap (\mathcal{N}_8(P))$ is 4-connected for every point $P \in S$.*

For example, the set of black points in Figure 5 is not 4-connected. So, "S is locally 4-connected" means that the critical configuration shown in Figure 5 cannot occur in the 8-neighborhood of any point of S.

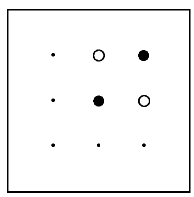


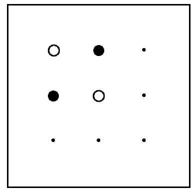
Figure 5: Critical configuration for non-well-composed sets.

Theorem 3.1 *S is well-composed iff S is locally 4-connected.*

The proof follows from Propositions 3.1 to 3.4 below.

Proposition 3.1 *If S is locally 4-connected, then $\overline{S}$ is locally 4-connected.*

Proof We show that if $\overline{S}$ is not locally 4-connected, then S is not locally 4-connected. If $\overline{S}$ is not locally 4-connected, then we can find a point $Q \in \overline{S}$ with an 8-neighborhood like the following (Q is the white point in the middle):



Then the 8-neighborhood of the point $N_4(Q) \in S$ (the black point on the left) is as in Figure 5. Hence S is not locally 4-connected. ∎

Proposition 3.2 *If S is locally 4-connected, then S is weakly well-composed.*

Proof If S is not well-composed, then a part of S is as in Figure 5; but then S is not locally 4-connected. ∎

Proposition 3.3 *If S is locally 4-connected, then S is well-composed.*

Proof This follows from Proposition 3.1 and Proposition 3.2 applied to S and $\overline{S}$. ∎

Proposition 3.4 *If S is well-composed, then S is locally 4-connected.*

Proof If S is not locally 4-connected, than we can find a point of S with an 8-neighborhood as in Figure 5. But in this case, either S or $\overline{S}$ is not weakly well-composed: If there is a 4-path in S connecting the two black points in Figure 5, then the two white points are 8-adjacent but not 4-connected. Hence $\overline{S}$ is not weakly well-composed. If there is no 4-path in S connecting these two black points, then S is not weakly well-composed. ∎

Remark The fact that S is weakly well-composed does not imply that S is locally 4-connected; see Figure 4.

Let us now note the following simple but important fact that 4-connectedness and 8-connectedness are equivalent for well-composed sets; this is an immediate consequence of the definition of well-composedness:

Proposition 3.5 *A well-composed set S is 4-connected iff S is 8-connected.*

Proposition 3.6 *Every 4-component of a well-composed set S is an 8-component of S and vice versa.*

Proof The vice versa part of this proposition is just the definition of well-composedness. Now if X is a 4-component of S, and there exists an 8-component Y of S such that X is a proper subset of Y, then the 8-component Y would not be a 4-component of S, a contradiction. $\blacksquare$

Definition 3.4 *Let X be any set in the plane. Let $\mathcal{Q}$ be a cover of the plane by closed squares with diameter d such that the intersection of two squares is either empty, a corner point or an edge. Such a cover is called a square grid with diameter d. A square of $\mathcal{Q}$ is black if its intersection with X is not empty, and white otherwise. If we treat the squares of $\mathcal{Q}$ as points of $\mathbb{Z}^2$ with the corresponding colors, we obtain a digital picture, which will be called the closed* digitization *of X with diameter d.*

If we consider only the interiors of the squares of cover $\mathcal{Q}$, i.e. a "cover" with open squares, then a digital picture obtained in this way will be called the open digitization of X with diameter d.

Proposition 3.7 *Let G be a set with the property that the intersection of G with every open ball with radius d is connected. Then*

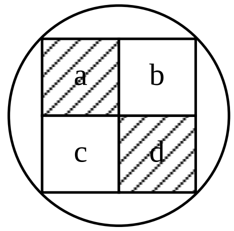
a) The closed digitization of G with diameter d is well-composed.

b) If G is open, the open digitization of G with diameter d is well-composed.

Proof a) Let a, b, c and d be four closed squares of cover $\mathcal{Q}$ sharing a common corner point x which are arranged as follows:

a	b
c	d

Let $B(x, d)$ be an open ball centered at x with radius d. We will show that the configurations $a \cap G \neq \emptyset$, $d \cap G \neq \emptyset$, and $c \cap G = b \cap G = \emptyset$ or $b \cap G \neq \emptyset$, $c \cap G \neq \emptyset$, and $a \cap G = d \cap G = \emptyset$, which lead to non-well-composedness of the digitization of G, are impossible. For example, if $a \cap G \neq \emptyset$, $d \cap G \neq \emptyset$, and $b \cap G = c \cap G = \emptyset$,



then $G \cap B(x,d)$ is disconnected, since $G \cap B(x,d) = G \cap (B(x,d) \setminus (c \cup d))$, which is clearly a disconnected subset of $B(x,d)$ (remember, squares c, d are closed).

b) Now let G be an open set, and let a, b, d and d be four open squares of cover $\mathcal{Q}$ which are arranged as above and such that their closures share a common corner point x. We will show that the configurations $a \cap G \neq \emptyset$, $d \cap G \neq \emptyset$, and $c \cap G = b \cap G = \emptyset$ or $c \cap G \neq$, $b \cap G \neq \emptyset$, and $a \cap G = d \cap G = \emptyset$, which lead to non-well-composedness of the digitization of G, are impossible. For example, if $a \cap G \neq \emptyset$, $d \cap G \neq \emptyset$, and $c \cap G = b \cap G = \emptyset$, then for $G \cap B(x,d)$ to be connected, we must have $x \in G$. Since G is open, there exists an open ball $B(x,\varepsilon) \subseteq G$ for some $\varepsilon > 0$. But then $B(x,\varepsilon)$ intersects all four open squares, so that we obtain $b \cap G \neq \emptyset$, $c \cap G \neq \emptyset$, which contradicts our assumption. ∎

Of course, not every digital picture is well-composed; but as mentioned in the Introduction, if the digitization process is topology-preserving, the resulting digital pictures must be well-composed. It can also be shown that any digital picture can be transformed into a well-composed digital picture by adding (or removing) a few black points whose positions can be identified using local operations.

4 Jordan Curve Theorem and Euler Characteristic

The Jordan Curve Theorem holds for well-composed sets, i.e. if we consider only subsets of $\mathbb{Z}^2$ which are well-composed, then every simple closed curve is well-composed, and therefore we have no problems with the paradoxes presented in the introduction. Due to Definition 2.5, a well-composed, simple closed curve is always a 4-curve.

Theorem 4.1 (Jordan Curve Theorem) *The complement of a well-composed, simple closed curve always has exactly two components.*

Proof Let C be a well-composed, simple closed curve. Rosenfeld [15] proved that if we consider 8-adjacency for C and 4-adjacency for $\overline{C}$ (or vice versa), then $\overline{C}$ has exactly two components. Our Theorem follows easily from his theorem: Since C is well-composed, it is 4- as well as 8-connected (Proposition 3.5). Due to Proposition 3.6, every 4-component of $\overline{C}$ is also an 8-component and vice versa. Hence $\overline{C}$ has exactly two components.

We can also prove this theorem directly following Rosenfeld's proof [15, 16], which is based on a standard proof of the theorem for polygons. We sketch only the main parts of it here: Let C be a well-composed, simple closed curve, and $P \notin C$; we say that $P = (x, y)$ is "inside" C if the half-line $H_p = \{(z, y) \mid x \leq z\}$ crosses C an odd number of times, and "outside" C otherwise (H_p may meet C in a sequence of consecutive points; such a sequence is a "crossing" if C enters the sequence from the row above H_p and exits to the row below H_p, or vice versa).

The main part of the proof is establishing that neighboring points of $\overline{C}$ are either both inside or both outside $\overline{C}$. This part can be very easily shown for well-composed sets. Hence points in the same component of $\overline{C}$ are either all inside or all outside. The theorem follows from this and the fact that the inside and outside of a curve are both nonempty. ∎

Kong and Rosenfeld [9] showed that if we use 4- (or 8-) connectedness for both a set and its complement, the Euler characteristic cannot be computed by counting local patterns. It is well known [12] that the Euler characteristic is locally computable if we use changeable 8/4- (or 4/8) connectedness. We will show that the Euler characteristic is also locally computable for well-composed sets.

Definition 4.1 *Let S be a digital set. If $\overline{S}$ has n_0 components and S has n_1 components, then $\gamma(S) = n_1 - n_0 + 1$ is called the* Euler characteristic *of S.*

Theorem 4.2 *The Euler characteristic is locally computable for well-composed sets.*

Proof Minsky and Papert [12, Chapter 5.8.1] proved this theorem using 4-adjacency for black points and 8-adjacency for white points. We show that our theorem follows easily from their theorem.

Let S be a well-composed set. Then every 4-component of S is an 8-component of S and vice versa (Prop. 3.6). The same holds for components of $\overline{S}$. Therefore, the theorem holds for every well-composed set.

We can also prove this theorem directly by following their proof. We then have even fewer cases to consider, since some of the sets treated by Minsky and Papert are not well-composed. ∎

5 Thinning

Rosenfeld [17] stated three requirements that a thinning algorithm should satisfy:

(α) Connectedness is preserved, for both the objects and their complement.

(β) Curves, arcs, and isolated points remain unchanged.

(γ) Upright rectangles, whose length and width are both greater than 1, do not remain unchanged.

In this section we present a sequential algorithm, and in Section 8 a parallel algorithm, which fulfill these requirements. In addition, these algorithms preserve well-composedness and produce really thin sets (a concept which we will also define precisely).

Before we give a standard definition of a simple point [10, 15], we want to remind the reader that when we use 8-connectedness for the foreground, and 4-connectedness for the background, we speak of 8-simple points, and in the opposite case we speak of 4-simple points.

Definition 5.1 *A point $P \in S$ is n-simple if it is a boundary point and $\mathcal{N}_n^*(P)$ contains just one n-component of S which is n-adjacent to P.*

Simple points are used in thinning algorithms:

Definition 5.2 Thinning *a digital set means repeated removal of simple points, but not endpoints (see Definition 2.5), from it.*

A digital set is termed n-irreducible *if its only n-simple points are n-endpoints.*

An irreducible set obtained from a set by means of thinning is called its skeleton.

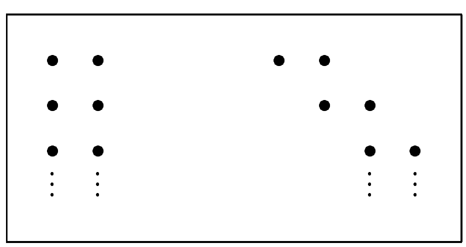
One step in a sequential thinning algorithm consists of removal of a single simple point. One step in a parallel thinning algorithm consists of the simultaneous removal of some set of simple points.

Remark Usually it is required that after application of a thinning algorithm the connectedness of a set and of its complement are not changed (Rosenfeld's condition (α)). This is usually easy to prove for sequential thinning [10, 15], but more difficult for parallel thinning.

The special treatment of endpoints guarantees that Rosenfeld's condition (β) is fulfilled.

There exists a simple local characterization of endpoints. 8-endpoints are characterized by the fact that they have exactly one 8-neighbor. There exist two different types of 4-endpoints which are endpoints in well-composed sets, namely those having exactly one neighbor which is a 4-neighbor and configuration 3 in Figure 6 (both of these configurations can occur as endpoints of 4-arcs).

The concept of not allowing deletion of 4-endpoints is justified by the fact that if we allow their deletion, the following "spikes" (see [21]) are reduced to two-point sets by any sequential thinning method that proceeds in a TV-scan-like sequence (row by row from top to bottom). On the other hand, when we fail to delete such "spikes", the skeleton may have "spurious" branches.



Note that the "spike" on the right remains unchanged (as it should be, since it is a 4-arc). The "spike" on the left should be reduced, by Rosenfeld's condition (γ). In fact, as can be easily seen, it becomes “thinner” but not “shorter”. If we want thinning to preserve well-composedness, only 4-simple points which are not endpoints can be deleted.

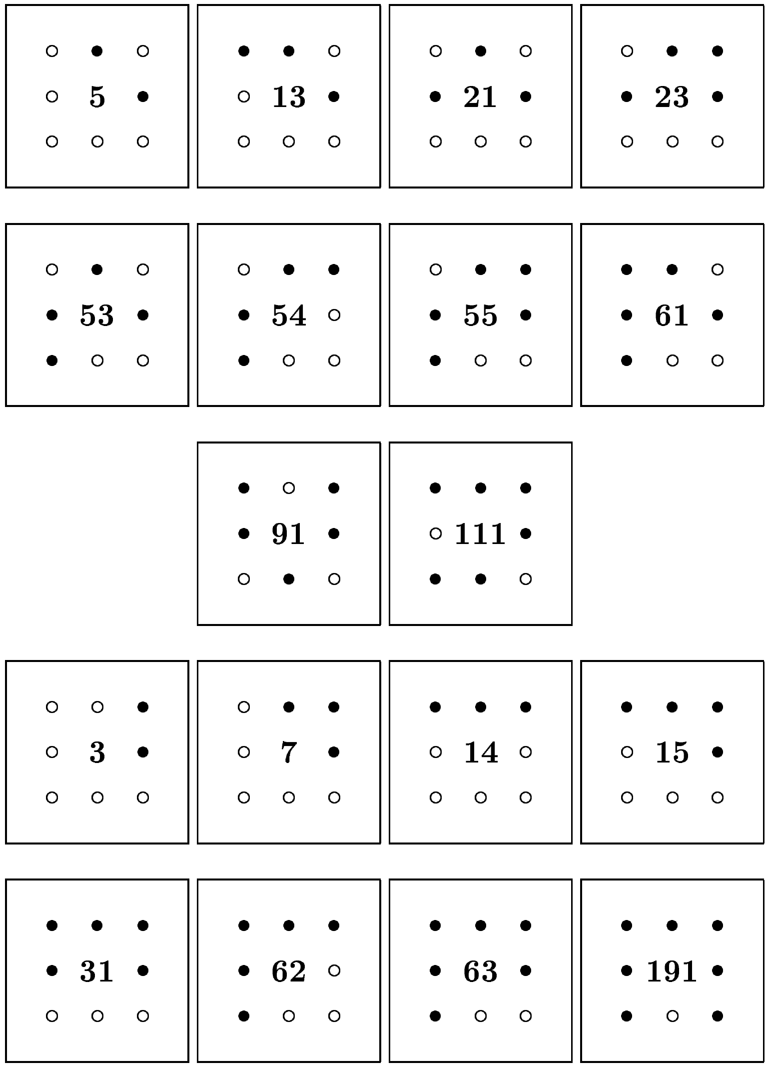
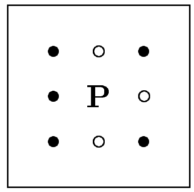


Figure 6: Configurations of all possible 8-simple points (except 8-endpoints). The middle point of configuration 3 is a 4-endpoint but not an 8-endpoint.

If we use changeable 8/4-connectedness, there are 18 types of 8-neighborhood configurations of 8-simple points (no including endpoint configurations), which generate 108 8-neighborhood configurations of 8-simple points by rotations and reflections ([6]). Figure 6 shows all 18 types of 8-simple points. An important advantage of dealing only with well-composed sets in thinning processes is the fact that we have to treat only 7 types of 4-simple point neighborhoods (without endpoints): 7, 14, 15, 31, 62, 63 and 191 (see Figure 6). This corresponds to 36 neighborhood configurations obtained by rotations and reflections. This fact makes thinning algorithms simpler.

The idea of deleting only 4-simple points in thinning an 8-connected object dates back to Rutovitz in 1966 [19] and was proposed by different authors ([11, 20, 22]); however, they did not use the concept of well-composed sets. In general, if we delete only 4-simple points to thin any subset of $\mathbb{Z}^2$, we have problems with 8-components, since 4-simple points are not necessarily 8-simple, as the following figure shows:



Obviously, at least one of the critical configurations $\begin{pmatrix} \circ & \bullet \\ \bullet & \circ \end{pmatrix}$ or $\begin{pmatrix} \bullet & \circ \\ \circ & \bullet \end{pmatrix}$ appears in such a situation. Thus, if we use well-composed sets, these problems cannot occur. In fact, in well-composed sets every 4-simple point is 8-simple.

Another very important advantage of well-composed sets is given in the following theorem. As already noted, in a well-composed set we can only have 4-simple points, since every component of a well-composed set is 4-connected. Therefore, thinning a well-composed set means removal of 4-simple points that are not endpoints.

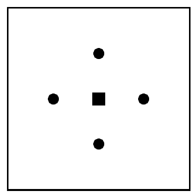
Theorem 5.1 *Sequential thinning is an internal operation on well-composed sets, i.e. applying sequential thinning to a well-composed set results in a well-composed set.*

Proof If we delete any (4-) simple point from a well-composed set, we obtain a well-composed set. To prove this, it is enough to show that if we can connect two points Q and R by a 4-path in a well-composed set X, then we can still connect Q and R by a 4-path after deleting any simple point P (different from Q and R) from X. The reason is that if we have any 4-path A in a well-composed set X passing through a simple point P, then the two direct neighbors of P in A belong to $\mathcal{N}_8^*(P)$. Since $\mathcal{N}_8^*(P) \cap X$ is 4-connected, we can connect these two direct neighbors by a path in $\mathcal{N}_8^*(P) \cap X$. Therefore, we can modify any path in X passing through a simple point P to a path with the same endpoints in $X \setminus P$. ∎

We conclude that thinning is an internal operation on well-composed sets if and only if only 4-simple points are deleted. It is easy to see that if we eliminate any 8-simple point which is not 4-simple, the resulting set will no longer be well-composed (see Figure 6).

One of the most important goals of thinning is to obtain a skeleton of the input set. Therefore, the resulting set, which cannot be further reduced by thinning, should not have any interior points. However, thinned irreducible sets can have many kernel components of arbitrarily large sizes as shown in [6]. The first example of a "large" kernel component was given by Arcelli in 1981 ([3]; see Figure 7).

For well-composed sets the situation is very simple, since there is only one type of kernel component, namely a set with only one point, as will be shown in Theorem 5.2. This type of interior point cannot be further eliminated, since it indicates a very useful property, namely that we have an intersection of two lines at this point, i.e. locally the following situation:



Eliminating such interior points would mean that a skeleton could not have such intersections of two line segments, which is an unrealistic assumption.

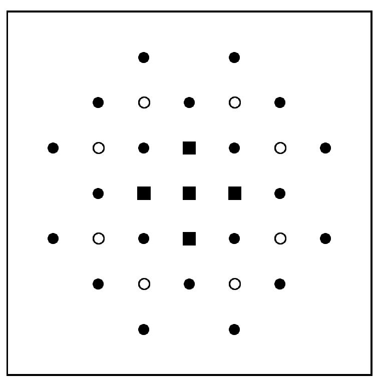


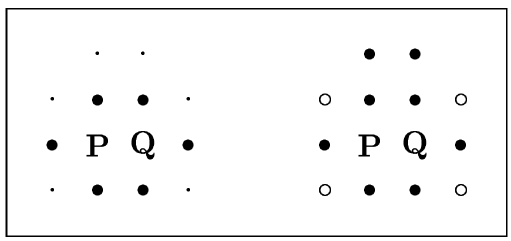
Figure 7: Arcelli's set with five interior points

Theorem 5.2 *Any (8-) connected component of the kernel of a 4-irreducible well-composed set contains at most one point.*

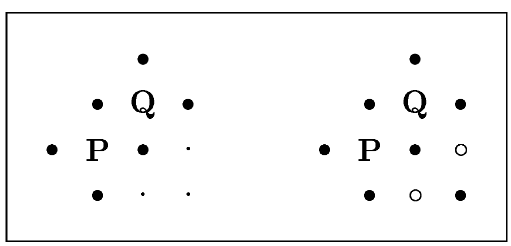
Proof We prove this by showing that it is not possible to have two adjacent interior points within a well-composed irreducible set.

Without loss of generality we may assume that we consider interior points having boundary points as direct neighbors. We distinguish two cases (in the pictures given in the proof, the one on the left gives the start situation and the one on the right gives the situation constructed during the proof):

a) P and Q are two directly neighboring interior points such that the points $N_2(P)$ and $N_2(Q)$ are not interior points. If $N_1(Q)$ were black then $N_2(Q)$ necessarily is a (4-) simple point or an interior point. So, $N_1(Q)$ must be white. Since $N_2(Q)$ cannot be simple and since the set is required to be well-composed, $N_2(N_2(Q))$ is black. The same argument holds for P. Now, $N_2(Q)$ becomes 4-simple, a contradiction.



b) Assume now that there are two indirectly neighboring interior points P and Q. Without loss of generality $Q = N_1(P)$ and $N_0(P)$ is not an interior point. This means that $N_0(N_0(P))$ or $N_7(P)$ is white. If one of these neighbors is black, then $N_0(P)$ is simple; hence they both are white. Now $N_0(N_7(P))$ must be black, for otherwise $N_0(P)$ would again be simple. The configuration thus obtained is no longer well-composed.



∎

Definition 5.3 *The* crossing number *(see e.g. [8, page 411]) is the number of white-black (0–1) transitions in the (cyclic) sequence $N_0(P)$, $N_1(P), \ldots, N_7(P)$, $N_0(P)$.*

Remark For well-composed sets, the crossing number is equal to the number of black 4-components in $\mathcal{N}_8^*(P)$, since the crossing number is equal to the number of black 4-components in $\mathcal{N}_8^*(P)$ if all 8-components in $\mathcal{N}_8^*(P)$ are directly connected to P. This is the case if the *critical configuration* $\begin{pmatrix} \circ & \bullet \\ \bullet & \circ \end{pmatrix}$ or $\begin{pmatrix} \bullet & \circ \\ \circ & \bullet \end{pmatrix}$ is not contained in $\mathcal{N}_8(P)$.

6 Irreducible Well-Composed Sets

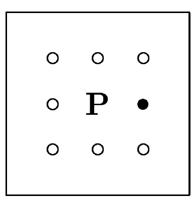
We now investigate sets having the property of being *(4-) irreducible* and well-composed. Such sets can be obtained by applying a (4-) thinning process to a well-composed set (see Theorem 5.1).

Irreducible sets obtained by ordinary thinning can contain all point configurations which are not simple. There are 148 such configurations (256 configurations of 3×3 neighborhoods -108 configurations of simple points that are not endpoints); they are generated by 33 neighborhood types. The situation is more favorable if thinning algorithms are applied to well-composed sets.

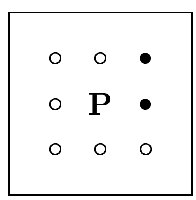
Theorem 6.1 *For a point P in an irreducible well-composed set, only the following neighborhood configurations (as well as symmetric configurations, obtained from them by 90° rotations and reflections) are possible:*

1. One direct black neighbor *(4-endpoint)*

a)

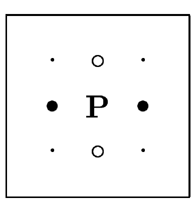


b)

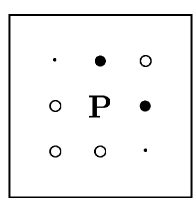


2. Two direct black neighbors

a)

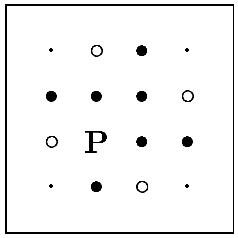


b)

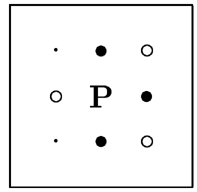


3. Three direct black neighbors

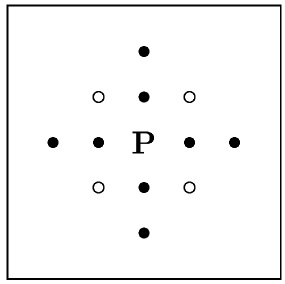
a)



b)



4. Four direct black neighbors



Proof The proof is by enumeration of all possible cases, showing that if P occurs in any other configuration, it must be a 4-simple point, which is not possible in an irreducible set.

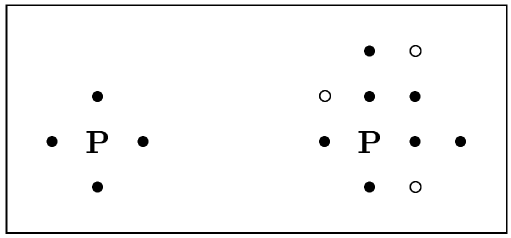
1. Let $N_0(P)$ be black. $N_3(P)$ and $N_5(P)$ cannot be black, since then the set would not be well-composed. If $N_1(P)$ were black, then P is a simple point and we have case 1b. The same holds for $N_7(P)$. If $N_1(P)$ and $N_7(P)$ are white, we have case 1a. Therefore, the two configurations shown above are the only possible configuration types with one direct black neighbor.

2. This case is obvious.

3. Let $N_0(P)$, $N_2(P)$, and $N_6(P)$ be black. If $N_1(P)$ and $N_7(P)$ are white, then we have case 3b. If only one of them is white, say $N_7(P)$, then we have case 3a. Since $N_0(P)$ cannot be simple, $N_0(N_0(P))$ must be black and $N_1(N_0(P))$ must be white. Since $N_1(P)$ cannot be simple, $N_2(N_1(P))$ must be black and $N_3(N_1(P))$ must be white.

4. In this case P is an interior point. Assume that $N_1(P)$ is black (see the picture below, where the set on the left represents the start situation, and the set on the right represents the situation constructed during the proof). If $N_7(P)$ were black, then $N_0(P)$ is either an interior point, which contradicts Theorem 5.2, or else it is necessarily simple. The same

argument applies to $N_3(P)$, so $N_3(P)$ and $N_7(P)$ must be white. $N_0(N_0(P))$ is not white, since otherwise either $N_0(P)$ would be simple ($N_7(N_0(P))$ white) or the set would not be well-composed ($N_7(N_0(P))$ black). Again, by symmetry, $N_2(N_2(P))$ is black. $N_2(N_1(P))$ must be white, since otherwise $N_1(P)$ would be simple. Now, regardless of the color of $N_0(N_1(P))$, $N_1(P)$ is always simple (since the set is assumed to be well-composed). It is now easily seen that the configuration is as shown above. ∎



Remark If P is an interior point of an irreducible well-composed set, then we have locally only the configuration presented in part 4 of the Theorem. So, P can be treated as an intersection point of a vertical and a horizontal line segment.

7 Graph Structure of Irreducible Sets

The goal of thinning is formulated by different authors as obtaining a set which is "one pixel thick" (*"... a single pixel wide ..."* in [13, p. 143], *"... until all that remains is lines which are one point wide ..."* in [8, p. 407], *"Thus, a final step might be necessary to reduce the set of the skeletal pixels to the unit width skeleton"* in [2, p. 411], *"A unitary skeleton is a single pixel thickness skeleton, in which each of its pixels is connected to not more than two adjacent pixels unless it represents a treepoint"* in [1, p. 13]), *"Overall, these applications employ thinning (a) to reduce line images to medial lines of unit width, (b) to enable objects to be represented as simplified data structures (e.g. by chain-coding) ..."* in [4].

Bearing in mind Arcelli's example (Figure 7), one might wonder how to give this requirement a precise meaning. Using the concept of well-composed sets, we can now propose the following definition:

Definition 7.1 *A digital set is* one pixel thick *if it is well-composed and (4-) irreducible.*

Thus we may give Theorem 5.2 an alternative formulation:

Theorem 7.1 *The skeleton of a well-composed set is one pixel thick.*

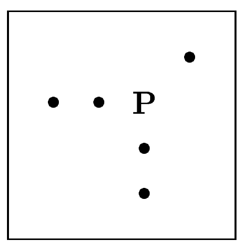
On the other hand, by "thinness" is meant intuitively that the skeleton should have a "graph-like" structure, as is expressed in the informal definition in [1] or [4]. Since any digital set which is equipped with a neighbor relation is a graph, we should make this concept more precise by formalizing the concept of a "graph-like structure".

Definition 7.2 *For any point P the* 8-connection number $C_8(P)$ *is the number of black 8-components in $\mathcal{N}_8^*(P)$ and the* 4-connection number $C_4(P)$ *is the number of black 4-components in $\mathcal{N}_8^*(P)$ which are directly connected to P. Obviously $C_8(P)$ is never greater than $C_4(P)$.*

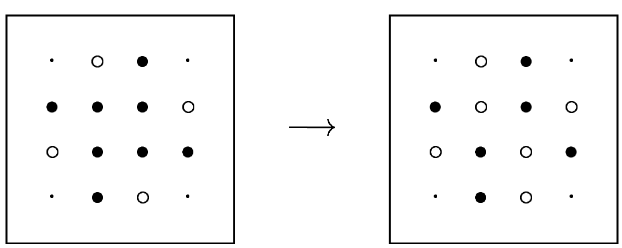
Definition 7.3 *A* graph point *of a digital set is a point having the property that either its 4-connection number equals the number of its black 4-neighbors or, if this is not the case, its 8-connection number equals the number of its black 8-neighbors.*

A digital set is termed a graph *if all of its points are graph points.*

The 8-skeleton of a digital set is not necessarily an 8-graph. The following set is 8-irreducible, but P is not an 8-graph point:



The same negative assertion holds for the 4-skeleton. In configuration 3a of Theorem 6.1, point P is not a 4-graph point. However, if this configuration is 8-thinned, we obtain the following configuration which consists entirely of 8-graph points:

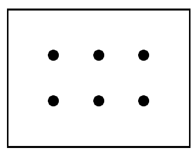


We define a point of a digital set to be a *4/8-graph point* if it is either a 4-graph point or an 8-graph point. A *4/8-graph* is a digital set consisting entirely of 4/8-graph points. Now we can formulate the ideas of the last part of this section in the following Theorem:

Theorem 7.2 *If a (4-) irreducible well-composed set is postprocessed by 8-thinning applied to all configurations of type 3a of Theorem 6.1, the resulting set is a 4/8-graph.*

8 Parallel Thinning on Well-Composed Sets

In sequential thinning, where only one simple point is deleted at each step, connectivity is evidently preserved. Parallel thinning, on the other hand, may not preserve connectivity. For example, if the points in the central column of the following set are deleted simultaneously, the set becomes disconnected. In order for parallel thinning algorithms to be topologically correct, one must avoid such situations.



When we investigate parallel thinning methods on well-composed sets, we are faced with an additional dilemma. Parallel elimination of points, even if it is designed so as to be topologically correct, may destroy well-composedness of sets, as shown by the example in Figure 8. Here simultaneous elimination of two 4-simple points yields a set which is not well-composed.

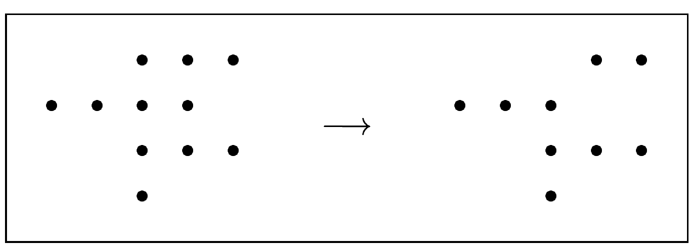


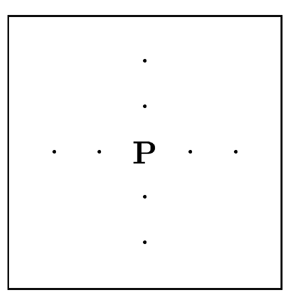
Figure 8: Parallel 4-thinning is not necessarily an internal operation on well-composed sets.

Our goal in this section is to construct a parallel thinning algorithm for well-composed sets that is topologically correct and also preserves well-composedness. We also want the resulting skeletons to be "thin", i.e. we want their kernel components to be as small as possible, as is the case for sequential thinning of well-composed sets. The simplest possibility is to use a 4-phase thinning algorithm as described by Rosenfeld in [17]. This algorithm removes only one type of border point at each phase: north border points are removed in the first phase, east in the second, and then south and west border points (where, for example, a north border point of S is one whose second neighbor is in $\overline{S}$). Rosenfeld showed that this algorithm is topologically correct. If we allow only 4-simple points to be deleted, this algorithm also preserves well-composedness, and the resulting skeletons are 4-irreducible. Therefore, Theorem 5.2 also holds for this 4-phase thinning. However, 4-phase algorithms are not very efficient, since only $1/4$ of the possible points can be deleted in one phase. It is also possible to thin well-composed sets using one-phase parallel algorithms. However, such algorithms must examine a relatively large neighborhood of every point, since it is well known [17] that a parallel thinning algorithm based on a 3×3 neighborhood cannot preserve connectedness.

We will now present a parallel thinning method for well-composed sets which minimizes the number of phases while at the same time minimizing the size of the neighborhoods used. It is a two-phase method consisting of one marking phase and one elimination phase. In the first phase the candidates for elimination are marked, and in the second phase these candidates are eliminated if they fulfill a simple condition described below.

Definition 8.1 *The* cross neighborhood *of a point P is*

$$\mathcal{CN}(P) = \mathcal{N}_4(P) \cup \{N_i(N_i(P)) \mid i = 0, 2, 4 \textit{ and } 6\}.$$

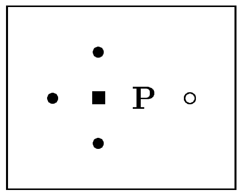


In [6] the concept of a *perfect point* was introduced.

Definition 8.2 *A point P of a digital set is termed* perfect *if*

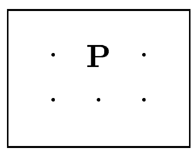
- *P has a direct neighbor which is an interior point, say $N_{2k}(P)$.*
- *The direct neighbor of P which is opposite the interior point $N_{2k+4 \pmod 8}(P)$ is white.*

For example, if P in the following set is black, then it is perfect.



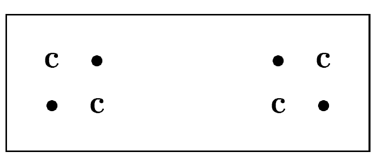
Definition 8.3 *The* south neighborhood *of a point P is*

$$\mathcal{SN}(P) = \{P, N_4(P), N_5(P), N_6(P), N_7(P), N_0(P)\}.$$



In a similar way the north neighborhood *of P can be defined.*

In the first step of the following parallel thinning algorithm, we will use the cross neighborhood of P to mark direct neighbors of P that are perfect points as candidates for deletion. In the second step, only points marked c can be deleted. While deleting the marked points, we must take care that in the following situations, at most one of the points marked c can be deleted:



Parallel Thinning Algorithm

Let S be a set of black points to be thinned and let P be any point in S.

First Step Candidates for deletion are marked.

If P is an interior point of S (i.e. $\mathcal{N}_4(P) \subseteq S$) and $N_i(N_i(P))$ is white, then $N_i(P)$ is marked c (candidate for deletion) for $i = 0, 2, 4$ and 6.

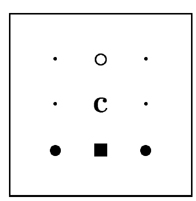
Second Step Deletion of marked points.

If a point P is marked c and no critical configuration (see below) occurs in the south neighborhood of P, then P is deleted (its color will be changed from black to white).

In the first step exactly the perfect points are marked as candidates. These are automatically simple points in a well-composed set, as the following Proposition states.

Proposition 8.1 *Every candidate point in a well-composed set is a simple point.*

Proof Since the point is a candidate point, we have the following situation (possibly rotated by a multiple of 90°):



Whatever color the points marked $\cdot$ have, the candidate point c is obviously simple, since the set is well-composed. ∎

Since every candidate is a perfect point, we are also guaranteed that the spikes shown in Section 5 will be left unchanged. Note also that endpoints are preserved from deletion, since points which are deleted are simple and perfect, and a perfect point has a direct neighbor which is an interior point.

The condition in the second step of this algorithm prevents deletion of a candidate if there is a critical configuration involving another candidate to the south. However, it cannot be that the deletion of every candidate will be prevented by this condition, since there always is a candidate having no critical configuration to the south. Therefore, the number of interior points decreases after every application of this algorithm. Hence, after we finish applying the algorithm, there are no (4-) simple perfect points in the resulting set. So, the kernel components of the resulting skeleton are exactly characterized by Theorem 8.3, which will be presented below. First, however, we prove that the algorithm preserves both well-composedness and connectivity.

Theorem 8.1 *The two-step parallel thinning method described above is an internal operation on well-composed sets.*

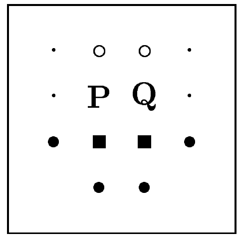
Proof Well-composedness is destroyed only when critical configurations occur and both candidates are deleted. This is prevented by deleting only candidates having no diagonal neighbors to the south which are also candidates. ∎

Theorem 8.2 *The two-step parallel thinning method is topologically correct on well-composed sets.*

Proof The proof is based on Ronse's conditions [14], of which a simplified version for well-composed sets is the following:

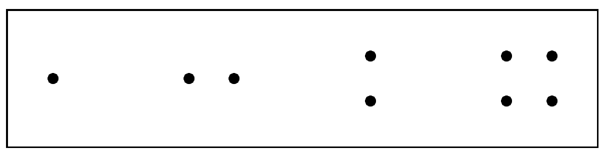
A parallel thinning algorithm on a well-composed set S preserves connectedness of the set and its complement if (1) only simple points are deleted and (2) for any two 8-adjacent points P and Q of S, P is simple after Q has been deleted, and Q is simple after P has been deleted. By Proposition 8.1, only simple points can be candidates, and therefore only simple points can be deleted. So it remains to show the second condition. Let P and Q be diagonal neighbors. Note that both points have to be simple and perfect in S in order to be deleted. In this case deletion of a diagonal neighbor cannot disconnect or delete a black 4-component in the 8-neighborhood of either P or Q. Therefore, P will be simple in $S \setminus \{Q\}$, and the same holds with P and Q interchanged.

Assume that P and Q are both candidates and also direct neighbors of each other. The only possibility for such a situation is indicated in the picture below (up to rotations by multiples of 90°).



If, for example, $N_6(Q)$ is an interior point and $N_2(Q)$ is white, then $N_6(P)$ must be an interior point and $N_2(P)$ must be white. It is now easy to see that P will be simple in $S \setminus \{Q\}$, and that the same holds with P and Q interchanged. ∎

Theorem 8.3 *The kernel of a well-composed set which contains no 4-simple perfect points is either empty or consists of 8-isolated components having one of the following forms:*

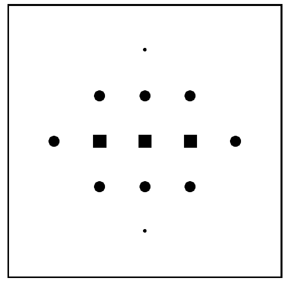


Proof The proof follows from Propositions 8.2 and 8.3 below: In a horizontal or vertical line there can at most be two adjacent kernel-boundary points (see Definition 8.4). The configuration in which the kernel contains two successive kernel-boundary points in a diagonal line such that one of their two common direct neighbors is not an interior point is impossible. Thus if the kernel contains two successive kernel-boundary points in a diagonal line, then their two common direct neighbors are also interior points. In this case the four kernel points form a square. ∎

Definition 8.4 *A point in the kernel of a set is termed* a kernel-boundary point *if it has at least one direct neighbor which is not in the kernel.*

Proposition 8.2 *If the kernel of S contains three successive kernel-boundary points in a horizontal or vertical line then S contains a point which is (4- and 8-) simple and perfect.*

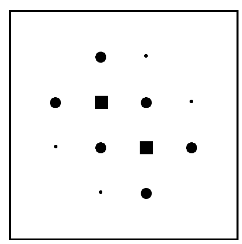
Proof We have the following situation:



At least one of the points marked $\cdot$ is necessarily white since otherwise the middle point would not be a kernel-boundary point. ∎

Proposition 8.3 *If the kernel of S contains two successive kernel-boundary points in a diagonal line, and one of their two common direct neighbors is not an interior point, then either S is not well-composed or it contains a point which is (4- and 8-) simple and perfect.*

Proof We have the following situation:



At least one of the points marked $\cdot$ must be white. Without loss of generality assume that the lower $\cdot$, call it Q, is white. If $N_3(Q)$ were black, then $N_2(Q)$ would be simple; hence both Q and $N_3(Q)$ are white. If $N_4(Q)$ were black, the set would not be well-composed; if it were white, $N_2(Q)$ would be simple. ∎

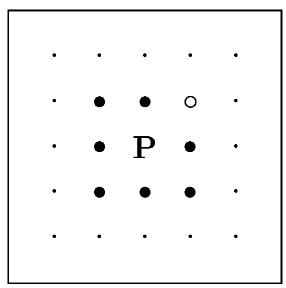
If the parallel thinning method described above is applied repeatedly to a well-composed set, the final remaining set does not contain any points which are simple and perfect. This set might contain kernel components as in Theorem 8.3. We will now show that the application of a sequential 4-thinning process to this remining set results in a 4-irreducible well-composed set which satisfies Theorem 5.2 and which automatically has all the properties of Sections 6 and 7. In this sequential thinning process each point in the set is examined once, and if it is simple, it is eliminated.

Theorem 8.4 *Assume that a well-composed set S contains no points which are simple and perfect. If the sequential 4-thinning process is applied to S, examining each point of S just once, the resulting set is 4-irreducible.*

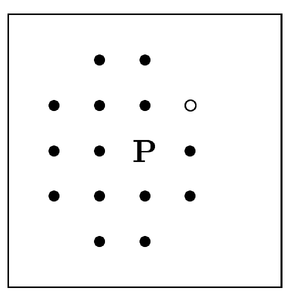
Proof There can be points of S which are simple but not perfect. We will show that after the sequential 4-thinning process is applied to S, no 4-simple points remain in the resulting set (which is also well-composed). It is clear that after the sequential 4-thinning process is applied to S, all simple points in S either have been deleted or are no longer simple. We now have to show that no point in the resulting set can become simple. More precisely, we have to show that the following situation is impossible: There is a point $P \in S$ that is not 4-simple in S, and there is a set $SP \subset \mathcal{N}_8(P)$ of 4-simple points in S such that P is 4-simple in $S \setminus SP$.

We distinguish two cases:

1. The 4-connection number $C_4(P)$ is equal to 1 with respect to S. In this case, P has at most one white indirect neighbor and since P is not simple in S, it must be an interior point. Therefore we have (up to 90° rotations) the following situation:



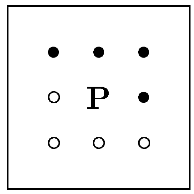
All points $N_{2i}(N_{2i}(P))$, $i = 0, 1, 2, 3$, must be black, since otherwise $N_{2i}(P)$ would be simple and perfect. Thus, $N_{2i}(P)$, $i = 2, 3$, are interior points. By the same argument $N_2(N_3(P))$, $N_4(N_3(P))$, $N_4(N_5(P))$, and $N_6(N_5(P))$ are black. Thus we have a kernel component containing five points (P and $N_i(P)$, $i = 3, 4, 5, 6,$), which contradicts Theorem 8.3.



2. $C_4(P) > 1$ with respect to S. The point P should be 4-simple with respect to the set $S \setminus SP$. Therefore, it has in this set one of the configurations 7, 14, 15, 31, 62, 63 or 191 in Figure 6. Configurations 63 and 191, however, are not possible since $C_4(P) > 1$ with respect to S. By the same argument configuration 62 is not possible since the set S is well-composed. To obtain configuration 31, SP consists of only one point, the 6-neighbor of P. Since this latter point must be simple, it would have to be perfect, a contradiction.

There remain only configurations 7, 14 and 15.

We start with configuration 15.



Since $C_4(P) > 1$, $N_6(P)$ and possibly $N_5(P)$ are black in S, and the points $N_4(P)$ and $N_7(P)$ are necessarily white.

If $N_5(P)$ is black, then $N_6(P)$ is not 4-simple. As a consequence, $N_6(P)$ must be eliminated as the last point in SP. For $N_6(P)$ to be eliminated, it must be simple in $S \setminus \{N_5(P)\}$. But then $N_6(P)$ is an endpoint (configuration 3).

A similar argument applies to configuration 14 with point $N_6(P)$ as the last point in SP to be eliminated. In case of configuration 7, we can apply the same argument either to $N_6(P)$ or to $N_4(P)$ as the last point in SP. ∎

9 Conclusions

When we restrict our attention to well-composed sets, a number of very difficult problems in digital geometry as well as complicated algorithmic approaches become extremely simple. On the other hand, if a set lacks the property of being well-composed, the digitization process that gave rise to it must not have been topology preserving. Since well-composedness is a local property, i.e. it depends on the colors of single picture points, it can be decided very efficiently in parallel whether a given set is well composed. If it is not, the set can be "repaired" by adding (or subtracting) single points.

We have investigated the process of thinning as an example of the usefulness of the concept of well-composedness. We saw that the thinning process (sequential as well as parallel) is greatly simplified and also that the resulting skeletons have a very simple structure if the input set is well-composed. These properties seem to give some intuitive concepts in the literature a rigorous meaning. Restricting thinning to well-composed sets leads to a very interesting situation: Although we delete points in fewer configurations, the resulting skeletons are thinner.

Further investigations are in progress. At present, we are studying the identification of sets having certain favorable (e.g. topological) properties with regard to picture processing algorithms.

Finally, the extension of these concepts and results to 3D images is being treated. The advantages of the concept of well-composedness turn out to be even greater in 3D.

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